Elzaki Decomposition Method for Solving Epidemic Model

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Abstract**—***This study investigate the application of Elzaki Decomposition Method in finding the approximate solution to the problem of the spread of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic. Epidemic models are nonlinear system of ordinary differential equation that has no analytic solution. The series solutions obtained by Elzaki Decomposition are compared with the existing results in the literatures; likewise, some plots were presented.The obtained results validate the efficiency of the method.*

Keywords—Adomian Decomposition, Approximate Solution, Epidemic Model,Elzaki transform, Non-fatal diseases.

I. INTRODUCTION

Epidemiology is the study of the distribution and determinants of health-related states or events (including disease)

Epidemic models of infectious diseases go back to Daniel Bernoulli's mathematical analysis of smallpox in 1760 [1], Where,

$x(t)$ implies susceptible population: those so far uninfected and therefore liable to infection;

$y(t)$ implies infective population: those who have the disease and are still at large;

$z(t)$ implies isolated population, or who have recovered and are therefore immune

β implies rate of change of susceptible population to infective population

γ implies rate of change of infective population to immune population

In this study, Elzaki transform is applied to obtain solution of an epidemic model which is assumed to have constant size over the period of the epidemic. Elzaki transform method [11-14], is a powerful tool for solving some differential and integral equation, and linear system of differential and integral equation.This new integral transform is a technique that used to solve linear differential equations and integral equations, but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms.

Elzaki transform operator is denoted by E[.] and is been defined by the integral equation as:

$$E[f(t),v] = T(v) = v\int_0^\infty e^{-\frac{t}{v}} f(t)\,dt, \qquad t>0 \qquad (2)$$

Let $T'(v)$ be Elzaki transform of the derivative of $f(t)$. Then:

and they have been developed extensively since the early 1900s. Hundreds of mathematical models have been published since, exploring the effects of bacterial, parasitic, and viral pathogens on human populations [2-5].

There have been many advances in disease management that have come from mathematical modelling.

Consider the problem of spreading of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic as considered in [6-10].

$$\frac{dx(t)}{dt} = -\beta x(t)y(t)$$
$$\frac{dy(t)}{dt} = \beta x(t)y(t) - \gamma y(t) \qquad (1)$$
$$\frac{dz(t)}{dt} = \gamma y(t)$$

With the initial conditions

$$x(0) = N_1, y(0) = N_2, z(t) = N_3$$

$$(i) \qquad T'(v) = \frac{T(v)}{v} - vf(0) \qquad (3)$$

$$(ii) \qquad T^{(n)}(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n-k} f^{(k)}(0), \quad n \geq 1 \qquad (4)$$

Adomian decomposition method can be easily applied to many linear and nonlinear problems in sciences and engineering fields [15-16]. The motivation of this paper is to extend the analysis of the modified Elzaki transformation method called Elzaki Decomposition Method (EDM) to solve the epidemic problem and compare the result with the one obtained in [6-10].

II. ANALYSIS OF ELZAKI DECOMPOSITION METHOD

Consider a general nonlinear differential equation

$$LU(x) + RU(x) + NU(x) = g(x) \qquad (5)$$

where L is the highest order linear differential operator, R is the linear differential operator of order less than L, N is the nonlinear differential operator, U is the dependent variable, x is an independent variable and g(x) is the source term.
Apply Elzaki Transform to equation (5) to have

$$E[LU(x)] + E[RU(x)] + E[NU(x)] = E[g(x)] \qquad (6)$$

using the differentiation property of the Elzaki transform (4) in (6) to have

$$\frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n-k} f^{(k)}(0) + E[RU(x)] + E[NU(x)] = E[g(x)] \qquad (7)$$

further simplification of (7) gave

$$T(v) - v^n \sum_{k=0}^{n-1} v^{2-n-k} f^{(k)}(0) + v^n \left[E[RU(x)] + E[NU(x)] - E[g(x)] \right] = 0 \qquad (8)$$

where E denotes the Elzaki transform,
application of Elzaki inverse Transform on (8) yielded

$$U(x) = G(x) - E^{-1}\left[v^n \left[[E[RU(x)] + S[NU(x)]] \right] \right] \qquad (9)$$

Where G(x) represents the term arising from the source term and the prescribed initial conditions.
The representation of the solution (9) as an infinite series is given below.

$$U(x) = \sum_{n=0}^{\infty} U_n(x) \qquad (10)$$

The nonlinear term is been decomposed as:

$$NU(x) = \sum_{r=0}^{\infty} A_n(u_0, u_1, \dots u_n) \qquad (11)$$

where A_n are the Adomian polynomials of functions $U_0, U_1, U_2 \dots U_n$ and can be calculated by formula given in [17] as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{r=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \quad n = 0,1,2,... \tag{12}$$

substituting (10) and (11) into (9) yielded

$$\sum_{n=o}^{\infty} U_{n+1}(x) = G(x) - E^{-1}\left[Ev^n\left[[[R\sum_{n=o}^{\infty}U_n(x)] + [\sum_{n=o}^{\infty}A_n]]\right]\right] \tag{13}$$

where

$$U_0(x) = G(x) = E^{-1}\left[\left[v^n\sum_{k=0}^{n-1}v^{2-n-k}U^{(k)}(0) + v^n E[g(x)]\right]\right] \tag{14}$$

and

$$\left. U_{n+1}(x) = -E^{-1}\left[v^n E\left[[[RU_n(x)] + [A_n]]\right]\right] \right\} \quad n \geq 0 \tag{15}$$

when the Elzaki Transform and the Elzaki inverse Transform are applied on (15) respectively, the iteration U_1, $U_2 \ldots U_n$ were obtained, which in turn gave the general solution as

$$U(x) = U_0(x) + U_1(x) + U_2(x) + U_3(x) + ... \tag{16}$$

III. NUMERICAL APPLICATION

In order to have a direction comparison with [6-10], the following parameters are used for (1)

$$x(0) = 20, y(0) = 15, z(0) = 10, \beta = 0.01, \gamma = 0.02 .$$

Following the procedure enumerated in section 2, the series solution and the graph for the susceptible population, infective population and isolated populationare respectively presented as:

$$x(t) = 20 - 3.0t - 0.045000t^2 + 0.0280500t^3 + 0.00079537500t^4 - 0.0003165502500t^5 - 0.0000125312512546t^6 + 0.000003596014977t^7 + ... \tag{17}$$

$$y(t) = 15 + 2.7t + 0.018000t^2 - 0.0281700t^3 - 0.000654525000t^4 + 0.0003191683500t^5 + 0.00001146735675t^6 - 0.000003628778853t^7 + ... \tag{18}$$

$$z(t) = 10 + 0.30t + 0.027000t^2 + 0.00012000t^3 - 0.000140850000t^4 - 0.00000261810000t^5 - 0.000001063894500t^6 + 3.276387642*10^{-8}t^7 + ... \tag{19}$$

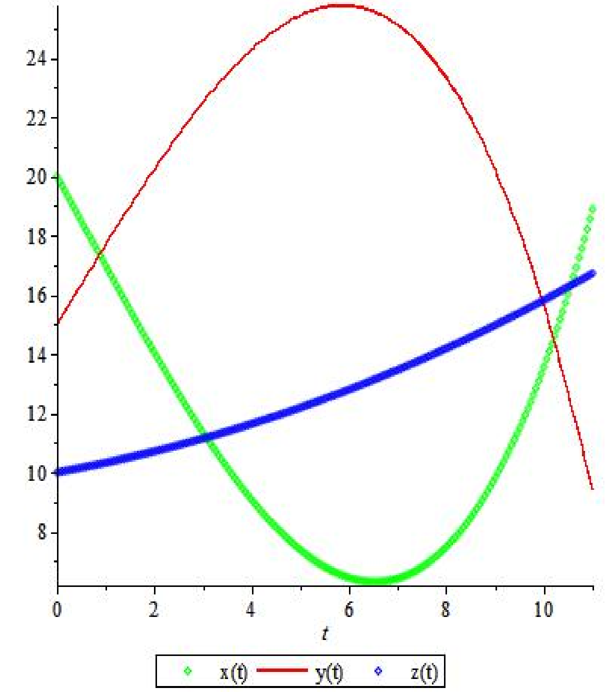

Fig.1: plot of three terms approximation for x(t), y(t) and z(t) versus time

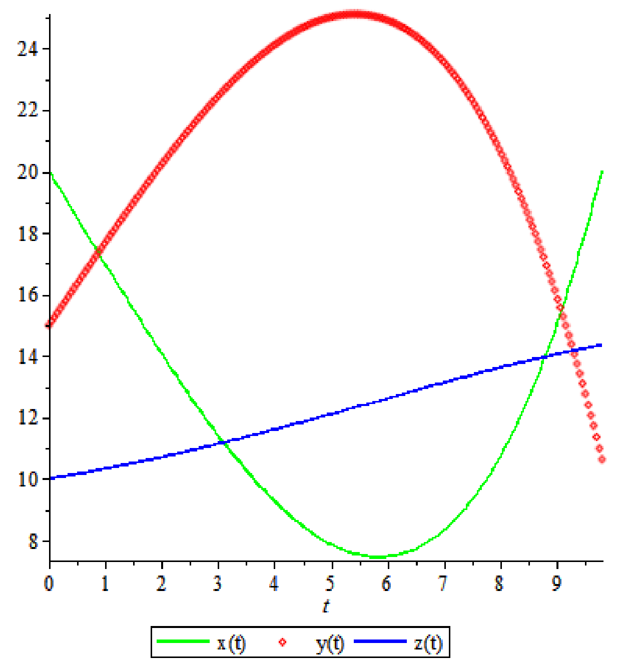

Fig.2: plot of four terms approximation for x(t), y(t) and z(t) versus time

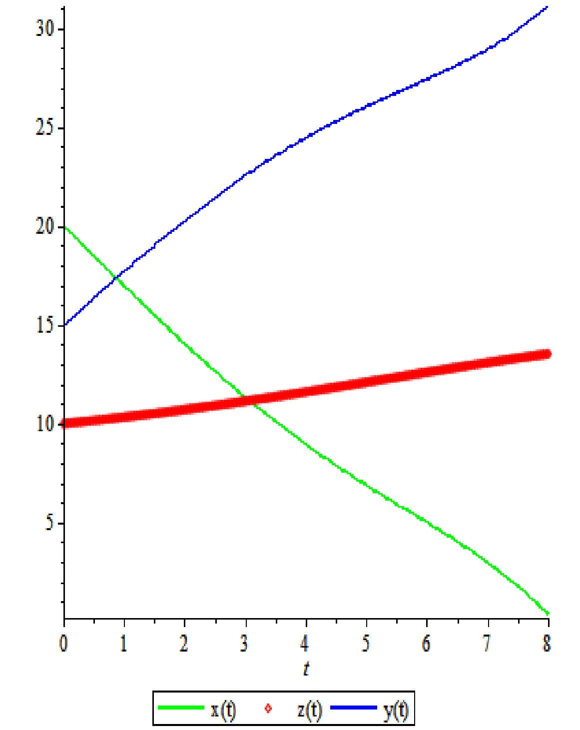

Fig.3: plot of five terms approximation for x(t), y(t) and z(t) versus time

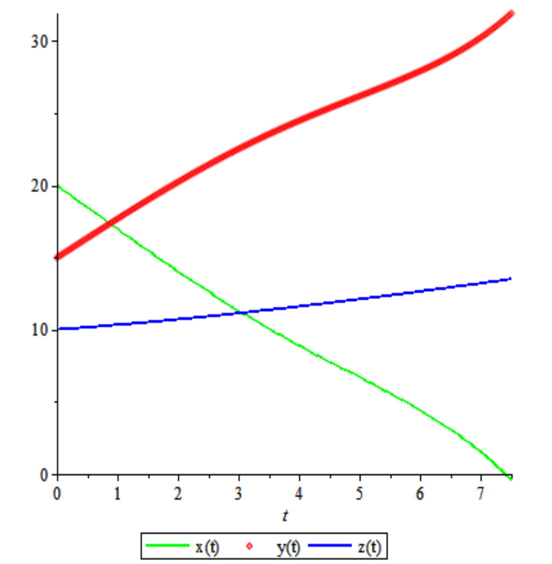

Fig.4: plot of six terms approximation for x(t), y(t) and z(t) versus time

The series solution and the graphical presentation above are in perfect agreement with the results obtained in [10-14], and from the graph it can be seen clearly that during the period of epidemic with a constant population as the number of susceptible individual decreases so also the number of infectious individual increases and vice versa while the number of immune individual increases.

IV. CONCLUSION

In this study, Elzaki Decomposition Method has been successfully employed to obtain an accurate solution to the problem of epidemic model without any difficulty. The results obtained are in complete agreement with Adomian Decomposition Method, Homotopy Perturbation Method, Variational Iteration Method and Differential Transform Method, this show that EDM is efficient, effective and accurate.

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